Generation of spatiotemporal correlated noise in 1+1 dimensions

Arne Traulsen, Karen Lippert, and Ulrich Behn
Institut für Theoretische Physik, Universität Leipzig, Vor dem Hospitaltore 1, 04103 Leipzig, Germany
(Received 15 August 2003; published 26 February 2004)

We propose a generalization of the Ornstein-Uhlenbeck process in $1+1$ dimensions which is the product of a temporal Ornstein-Uhlenbeck process with a spatial one and has exponentially decaying autocorrelation. The generalized Langevin equation of the process, the corresponding Fokker-Planck equation, and a discrete integral algorithm for numerical simulation are given. The process is an alternative to a recently proposed spatiotemporal correlated model process [J. García-Ojalvo *et al.*, Phys. Rev. A **46**, 4670 (1992)] for which we calculate explicitly the hitherto unknown autocorrelation function in real space.

DOI: 10.1103/PhysRevE.69.026116 PACS number(s): 02.50.Ey, 05.40.−a, 05.10.Gg, 02.60.Cb

I. INTRODUCTION

Noise induced phenomena are subject of considerable recent attention [1,2]. After considering in the early phase systems with only few degrees of freedom [1] in the last decade the effects of noise in spatially distributed systems have been investigated [2]. In this context stochastic model processes are necessary to mimic spatiotemporal fluctuations of different origin. If characteristic time and length scales of the system and noise are clearly separated, the use of a spatiotemporal Gaussian white noise may be justified, but it can also lead to spurious results [3–5]. There are physical situations where the characteristic scales are not well separated, e.g., in externally driven systems [6], or where the square of the driving stochastic process is involved. Both hold in electrohydrodynamic convection in nematic liquid crystals (see, e.g., Ref. [7]), driven by external stochastic electric fields [8,9]. Further examples are the influence of spatiotemporal colored noise on spatiotemporal chaos modeled by the complex Ginzburg-Landau equation [10] and on networks of excitable systems displaying spatiotemporal stochastic resonance [11]. This sufficiently motivates to model correlated spatiotemporal fluctuations. For an approach based on different grounds see, e.g., Ref. [12].

A frequently used spatiotemporal correlated model process was introduced by García-Ojalvo, Sancho, and Ramirez-Piscina (GSR) [13] who considered in spatial dimension d, $\mathbf{r}\in R^d$ the stochastic partial differential equation (PDE)

$$\tau\frac{\partial}{\partial t}\varphi(\mathbf{r},t)=-(1-\lambda^2\Delta)\varphi(\mathbf{r},t)+\xi(\mathbf{r},t), \tag{1}$$

where the additive driving process $\xi(\mathbf{r},t)$ is Gaussian distributed with zero mean and with autocorrelation $K^{\xi}(\mathbf{r},t;\mathbf{r}',t')=\langle\xi(\mathbf{r},t)\xi(\mathbf{r}',t')\rangle=\sigma^2\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$.

The heuristics of Eq. (1) is evident: the diffusive term effectively reduces the lifetime of Fourier components with wavelengths short compared to λ, see also below. For $\lambda=0$ it reduces to the Langevin equation defining the common temporal Ornstein-Uhlenbeck process (OUP) [14–16]. The solutions are thus, in a sense, generalizations of the OUP. Equation (1) belongs to a class of stochastic PDEs for which existence and uniqueness of the solution are proven rigorously [17]. It is discussed also in the context of reaction-diffusion systems (see, e.g., Ref. [18]), and within a generating functional approach [12].

In this paper we propose an alternative spatiotemporal generalization of the OUP in $1+1$ dimension which is simply the product of a temporal OUP with a spatial one and has exponentially decaying autocorrelation. To make the paper self-contained and to introduce the notation that is used in the sequel, we shortly recall in Sec. II basic properties and numerical generation of the common OUP in one (temporal) dimension. The scaling necessary when transforming between discrete and continuous formulations is carefully discussed. In Sec. III the generalized OUP is constructed independently within a spatially discretized scheme and in a continuous version as the solution of a stochastic PDE different from Eq. (1). Subsequently, conditions that ensure stationarity and homogeneity are discussed, the generalized Fokker-Planck equation and its stationary solution are given, and numerically generated data are compared with the analytic results. In Sec. IV the autocorrelation function of the GSR process is explicitly calculated in real space for $d=1$ in both continuous and discrete formulations and compared with numerical results. Previous work studied the behavior in real space only for spatial dimensions $d=2$ [13,19] and $d=3$ [19]; see, however, Ref. [20]. Contrary to conventional outlook [10,22], the autocorrelations of the GSR process decay not exponentially but in a more intricate way.

Problems connected with the generalization to higher dimensions are briefly discussed in the concluding section.

II. THE ORNSTEIN-UHLENBECK PROCESS

The OUP is the only stationary Gaussian Markov process with exponentially decaying autocorrelation (Doob's theorem [23]). Realizations $\eta(t)$ of the OUP can be generated by solving the Langevin equation

$$\tau\frac{d}{dt}\eta(t)=-\eta(t)+\xi(t), \tag{2}$$

where $\xi(t)$ is a Gaussian white noise with $\langle\xi(t)\xi(t')\rangle=\sigma_t^2\delta(t-t')$. In a mathematically precise form, Eq. (2) reads

$$\tau d\,\eta(t) = -\,\eta(t)dt + dW(t), \tag{3}$$

where W is a Wiener process with $\langle W(t)W(t')\rangle = \sigma_t^2 \min(t,t')$; note $dW(t)/dt = \xi(t)$. Solving Eq. (3) with initial condition $\eta(t_0) = \eta_0$ gives

$$\eta(t) = \eta_0 e^{-(t-t_0)/\tau} + \frac{1}{\tau} e^{-t/\tau} \int_{t_0}^{t} dW(s) e^{s/\tau}, \tag{4}$$

which has the autocorrelation

$$\langle \eta(t)\,\eta(t')\rangle = \left(\langle \eta_0^2 \rangle - \frac{\sigma_t^2}{2\tau} \right) e^{-(t+t'-2t_0)/\tau} + \frac{\sigma_t^2}{2\tau} e^{-|t-t'|/\tau}. \tag{5}$$

The process becomes stationary if the initial values are Gaussian distributed with zero mean and variance $\sigma_t^2/2\tau$, or in the limit $t,t' \to \infty$, or for $t_0 \to -\infty$; it is then the OUP. We denote the stationary part of the autocorrelation function as

$$K^{\eta}(t-t') = \frac{\sigma_t^2}{2\tau} e^{-|t-t'|/\tau}. \tag{6}$$

Naturally, $K^{\eta}(t-t')$ solves the inhomogeneous equation which is obtained by multiplying Eq. (2) by $\eta(t)$ given by Eq. (4) and averaging

$$\tau \frac{d}{dt} K^{\eta}(t-t') = -K^{\eta}(t-t') + \Theta(t'-t) \frac{\sigma^2}{\tau} e^{-(t'-t)/\tau}. \tag{7}$$

Given the value $\eta(t)$ we can obtain $\eta(t+\Delta t)$ as

$$\eta(t+\Delta t) = \eta(t) e^{-\Delta t/\tau} + \frac{1}{\tau} e^{-(t+\Delta t)/\tau} \int_{t}^{t+\Delta t} dW(s) e^{s/\tau}, \tag{8}$$

where the last term on the right-hand side (rhs) is a stochastic increment. The increments in nonoverlapping time intervals are obviously independent; they have zero mean and the variance, see, e.g., Ref. [24],

$$\frac{\sigma_t^2}{2\tau}(1 - e^{-2\Delta t/\tau}). \tag{9}$$

Introducing the notation $\eta_t = \eta(t\Delta t)$, where Δt is fixed and $t = \ldots, -1, 0, 1, \ldots$, one obtains, for every choice of Δt, an exact recursion relation for equidistant discrete times (discrete integral algorithm),

$$\eta_{t+1} = \eta_t e^{-\Delta t/\tau} + \frac{\sigma_t}{\sqrt{2\tau}} \sqrt{1 - e^{-2\Delta t/\tau}}\, \tilde{\xi}_{t+1}, \tag{10}$$

where $\tilde{\xi}_t$ are independent zero mean Gaussian random numbers with variance 1, cf. Ref. [25].

For *small* Δt a Taylor expansion of the rhs of Eq. (8) leads to

$$\eta(t+\Delta t) = \eta(t)\left(1 - \frac{\Delta t}{\tau}\right) + \frac{\Delta t}{\tau} \xi(t+\Delta t). \tag{11}$$

The discrete version of Eq. (11) is obtained using the above notation and replacing $\xi(t)$ by $\sigma_t \tilde{\xi}_t / \sqrt{\Delta t}$. This ensures the correct autocorrelation in the continuum limit observing $\lim_{\Delta t \to 0} \delta_{t,t'}/\Delta t = \delta(t-t')$ and amounts to a rescaled variance $\tilde{\sigma}_t^2 = \sigma_t^2/\Delta t$,

$$\eta_{t+1} = \eta_t \left(1 - \frac{\Delta t}{\tau}\right) + \frac{\Delta t}{\tau} \tilde{\sigma}_t \tilde{\xi}_{t+1}. \tag{12}$$

All results are consistent: Eq. (11) can be derived from Eq. (2) using an Euler discretization, and Eq. (12) from Eq. (10) by a Taylor expansion of the coefficients.

III. A GENERALIZATION TO 1+1 DIMENSIONS

In this section we generalize the OUP and construct in 1+1 dimensions a spatiotemporal random field $\varphi(x,t)$ with reasonable properties. For fixed x the process should be the common temporal OUP described above and for fixed t a spatial OUP. It is reasonable to require translational invariance, analogous to temporal stationarity, of all averages and an exponential decay of the spatiotemporal autocorrelation

$$K^{\varphi}(x-x',t-t') = \langle \varphi(x,t)\varphi(x',t')\rangle = \frac{\sigma^2}{4\lambda\tau} e^{-|x-x'|/\lambda - |t-t'|/\tau}, \tag{13}$$

where $\sigma = \sigma_s \sigma_t$, σ_s and λ characterizing for fixed time the spatial process.

We propose two independent schemes leading to the same result. First we employ a spatially discrete scheme to construct more general spatiotemporal correlated processes given in Ref. [26]. Alternatively, we consider a linear stochastic PDE different from Eq. (1) driven by additive Gaussian spatiotemporal white noise and show that its stationary solutions are Gaussian distributed and have the desired properties. The analytic results are corroborated by numerical data.

A. Recursive generation

We consider the field $\varphi(x,t)$ on equidistant lattice sites i, $i = 0, \ldots, N$, adopting the notation $\varphi_i(t) = \varphi(i\Delta x, t)$. In the first step of construction we generate spatially independent OUPs $\eta_i(t)$ using a standard algorithm, e.g., Refs. [24,27], with autocorrelation

$$K_{ij}^{\eta}(t-t') = \langle \eta_i(t)\,\eta_j(t')\rangle = \frac{\sigma^2}{4\lambda\tau} e^{-|t-t'|/\tau} \delta_{i,j}. \tag{14}$$

We then, as proposed in Ref. [26] to construct a more general spatiotemporal correlated noise, superpose these processes

$$\varphi_i(t) = \sum_{k=0}^{i} a_{ik}\, \eta_k(t), \quad i = 0, \ldots, N. \tag{15}$$

Since this expression is linear in η_k, also the φ_i are Gaussian distributed with zero mean. Requiring that the spatial autocorrelation is the discrete version of Eq. (13) for equal times,

$$K_{ij}^{\varphi}(0)=\sum_{k=0}^{\min(i,j)} a_{ik}a_{jk}\langle \eta_k(t)^2\rangle=\frac{\sigma^2}{4\tau\lambda}e^{-|i-j|\Delta x/\lambda}, \quad (16)$$

determines the coefficients a_{jk}. It is easy to check that

$$a_{jk}=e^{-(j-k)\Delta x/\lambda}(\sqrt{1-e^{-2\Delta x/\lambda}})^{1-\delta_{k,0}}, \quad (17)$$

where $j=0,\ldots,N$ and $k=0,\ldots,j$. Using these coefficients, we write

$$\varphi_j(t)=e^{-j\Delta x/\lambda}\eta_0(t)+e^{-(j-1)\Delta x/\lambda}\sqrt{1-e^{-2\Delta x/\lambda}}\eta_1(t)+\cdots + \sqrt{1-e^{-2\Delta x/\lambda}}\eta_j(t). \quad (18)$$

With the corresponding formula for $\varphi_{j+1}(t)$, we find

$$\varphi_{j+1}(t)=e^{-\Delta x/\lambda}\varphi_j(t)+\sqrt{1-e^{-2\Delta x/\lambda}}\eta_{j+1}(t), \quad (19)$$

where $\eta_{j+1}(t)$ are the spatially independent random numbers specified above, each of which being a temporal OUP. Obviously, this is the spatial analog of the discrete integral algorithm (10) for the temporal process.

In discrete notation both for space and time we insert $\eta_{j+1,t+1}$ from Eq. (10), after replacing $\sigma_t/\sqrt{2\tau}\to\sigma/\sqrt{4\tau\lambda}$, into Eq. (19) (written for $\varphi_{j+1,t+1}$), and obtain finally

$$\varphi_{j+1,t+1}=e^{-\Delta x/\lambda}\varphi_{j,t+1}+e^{-\Delta t/\tau}\varphi_{j+1,t}-e^{-\Delta t/\tau-\Delta x/\lambda}\varphi_{j,t} + \sqrt{1-e^{-2\Delta x/\lambda}}\sqrt{1-e^{-2\Delta t/\tau}}\frac{\sigma}{\sqrt{4\tau\lambda}}\tilde{\xi}_{j+1,t+1}, \quad (20)$$

where $\langle\tilde{\xi}_{i,t}\tilde{\xi}_{j,t'}\rangle=\delta_{i,j}\delta_{t,t'}$. For simulations, this discrete integral algorithm is preferable since by construction it is correct for any choice of Δx and Δt.

Expanding the coefficients in Eq. (20) for small Δx and Δt, a first-order discrete differential algorithm, the generalization of Eq. (12), is obtained,

$$\varphi_{j+1,t+1}=\left(1-\frac{\Delta x}{\lambda}\right)\varphi_{j,t+1}+\left(1-\frac{\Delta t}{\tau}\right)\varphi_{j+1,t} -\left(1-\frac{\Delta x}{\lambda}\right)\left(1-\frac{\Delta t}{\tau}\right)\varphi_{j,t}+\frac{\Delta x\Delta t}{\lambda\tau}\tilde{\sigma}\tilde{\xi}_{j+1,t+1}, \quad (21)$$

where $\tilde{\sigma}=\sigma/\sqrt{\Delta x\Delta t}$. Writing Eq. (21) in continuous notation,

$$\varphi(x+\Delta x,t+\Delta t)=\left(1-\frac{\Delta x}{\lambda}\right)\varphi(x,t+\Delta t)+\left(1-\frac{\Delta t}{\tau}\right) \times\varphi(x+\Delta x,t)-\left(1-\frac{\Delta x}{\lambda}\right)\left(1-\frac{\Delta t}{\tau}\right) \times\varphi(x,t)+\frac{\Delta x\Delta t}{\lambda\tau}\xi(x+\Delta x,t+\Delta t), \quad (22)$$

we have replaced $\tilde{\sigma}\tilde{\xi}_{j,t}\to\xi(x,t)$ in complete analogy with the rescaling for the temporal OUP. The autocorrelation of the spatiotemporal Gaussian white noise is $\langle\xi(x,t)\xi(x',t')\rangle=\sigma^2\delta(x-x')\delta(t-t')$.

B. Continuous approach

1. Generalized Langevin equation

An alternative approach starts from a generalized Langevin equation in $1+1$ dimensions, the stochastic PDE

$$\left(1+\tau\frac{\partial}{\partial t}+\lambda\frac{\partial}{\partial x}+\lambda\tau\frac{\partial^2}{\partial x\partial t}\right)\varphi(x,t)=\xi(x,t), \quad (23)$$

which for $\lambda=0$ reproduces the Langevin equation (2) generating the temporal OUP and for $\tau=0$ that of the spatial OUP. The spatiotemporal Gaussian white noise can be conceived as the product $\xi(x,t)=\xi(x)\xi(t)$, where $\xi(x)$ and $\xi(t)$ denote independent spatial and temporal Gaussian white noise, respectively.

It is interesting to note that Eq. (23) is a hyperbolic PDE, whereas Eq. (1) is a parabolic one. Equation (23) has the two families of characteristics $x=$ const and $t=$ const, the latter is the only family of characteristics of Eq. (1). Correspondingly, the solution of Eq. (1) reproduces in the limit $\lambda\to0$ the temporal OUP multiplied by $\delta(x-x')$ but $\tau\to0$ does not result in the spatial OUP; see below.

The stochastic PDE (23) can be obtained from the continuous differential algorithm (22) by Taylor expansion of φ for small Δx and Δt and performing the limit $\Delta x,\Delta t\to0$. Alternatively, it can be conceived as the product of the two Langevin equations for a temporal OUP, Eq. (2), and its spatial analog. For this we denote the product of the temporal and spatial OUPs by $\varphi(x,t)$ and observe that the differential operator on the lhs of Eq. (23) factorizes as $(1+\tau\partial/\partial t+\lambda\partial/\partial x+\lambda\tau\partial^2/\partial x\partial t)=(1+\tau\partial/\partial t)(1+\lambda\partial/\partial x)$.

Using a separation ansatz, a solution of Eq. (23) can be written as

$$\varphi(x,t)=f(x)g(t), \quad (24)$$

where

$$f(x)=f_0e^{-(x-x_0)/\lambda}+\frac{A}{\lambda}e^{-x/\lambda}\int_{x_0}^{x}dW(y)e^{y/\lambda}, \quad (25)$$

$$g(t)=g_0e^{-(t-t_0)/\tau}+\frac{1}{A\tau}e^{-t/\tau}\int_{t_0}^{t}dW(s)e^{s/\tau}, \quad (26)$$

and $f(x_0)=f_0$ and $g(t_0)=g_0$ denote boundary and initial values. The initial and boundary processes $\varphi(x,t_0)$ and $\varphi(x_0,t)$ are OUPs with correlation length λ and correlation time τ, respectively. Note the appearance of the extra factors A and $1/A$ in Eqs. (25) and (26) respectively, compared with the process given by Eq. (4). $0<|A|<\infty$ is an arbitrary constant that corresponds to the separation constant for a deterministic PDE. In the nonstationary case it weights the relative influence of the initial and boundary realizations. In the terms of $\varphi(x,t)$ which survive in the stationary case, A cancels and naturally its value plays no role, see below.

Exploiting that the spatial and temporal Wiener processes $W(y)$ and $W(s)$ are independent and have zero mean, we obtain the autocorrelation function

$$\begin{aligned} K^{\varphi}(x,t;x',t') = & \frac{\sigma^2}{4\tau\lambda} e^{-|x-x'|/\lambda-|t-t'|/\tau} \\ & + \frac{\sigma_t^2}{2\tau}\left(\langle f_0^2\rangle \frac{1}{A^2} - \frac{\sigma_s^2}{2\lambda}\right) e^{-(x+x'-2x_0)/\lambda-|t-t'|/\tau} \\ & + \frac{\sigma_s^2}{2\lambda}\left(\langle g_0^2\rangle A^2 - \frac{\sigma_t^2}{2\tau}\right) e^{-|x-x'|/\lambda-(t+t'-2t_0)/\tau} \\ & + \left(\langle f_0^2\rangle \frac{1}{A^2} - \frac{\sigma_s^2}{2\lambda}\right)\left(\langle g_0^2\rangle A^2 - \frac{\sigma_t^2}{2\tau}\right) \\ & \times e^{-(x+x'-2x_0)/\lambda-(t+t'-2t_0)/\tau}. \end{aligned} \tag{27}$$

The first term on the rhs is just the desired stationary and homogeneous autocorrelation, independent on the boundary and initial conditions, see Eq. (13). The remaining terms disappear for $x_0\to-\infty$ and $t_0\to-\infty$, respectively. A second possibility to make the nonstationary and nonhomogeneous terms vanish is to choose f_0 and g_0 as zero mean Gaussian distributed with variance such that

$$\langle f_0^2\rangle = \frac{\sigma_s^2}{2\lambda}A^2 \quad \text{and} \quad \langle g_0^2\rangle = \frac{\sigma_t^2}{2\tau}\frac{1}{A^2}. \tag{28}$$

In this case the process will be homogeneous and stationary from the beginning. The variance of the process φ is independent of (x,t), $\langle\varphi^2(x,t)\rangle=\langle f_0^2\rangle\langle g_0^2\rangle=\sigma^2/4\lambda\tau$.

2. Generalized Fokker-Planck equation

The Fokker-Planck equation (FPE) corresponding to a stochastic PDE should be a functional equation. For the spatially discretized system the FPE is a matrix equation. We will derive for this case the generalized FPE and its stationary solution. Discretizing Eq. (23) using a first-order Euler scheme gives the system of ordinary differential equations,

$$\tau\mathbf{C}d\boldsymbol{\varphi}(t) = -\mathbf{C}\boldsymbol{\varphi}(t)dt + \frac{\sigma_s}{\sqrt{\Delta x}}d\mathbf{W}(t), \tag{29}$$

where $\boldsymbol{\varphi}=(\varphi_1,\ldots,\varphi_N)^{\mathrm{T}}$ and $\mathbf{W}=(W_1,\ldots,W_N)^{\mathrm{T}}$. The matrix $\mathbf{C}$ has the nonvanishing elements

$$c_{i,i}=c_0=1+\frac{\lambda}{\Delta x}, \quad c_{i+1,i}=c_1=-\frac{\lambda}{\Delta x}. \tag{30}$$

Since $\det\mathbf{C}\neq 0$, we can multiply Eq. (29) by $\mathbf{C}^{-1}$ and obtain

$$\tau d\boldsymbol{\varphi}(t) = -\boldsymbol{\varphi}(t)dt + \frac{\sigma_s}{\sqrt{\Delta x}}\mathbf{C}^{-1}d\mathbf{W}(t). \tag{31}$$

Now we can treat the system as a multivariate OUP. It can be shown [18] that the corresponding FPE is

$$\begin{aligned} \frac{\partial}{\partial t}p &= -\sum_i \frac{\partial}{\partial\varphi_i}\sum_j\left[-\frac{1}{\tau}\delta_{ij}\varphi_j p \right. \\ &\quad \left. - \frac{\sigma_s^2\sigma_t^2}{2\tau^2\Delta x}(\mathbf{C}^{-1}(\mathbf{C}^{-1})^{\mathrm{T}})_{ij}\frac{\partial}{\partial\varphi_j}p\right] \\ &= -\sum_i \frac{\partial}{\partial\varphi_i}J_i = -\boldsymbol{\nabla}_{\varphi}\mathbf{J}, \end{aligned} \tag{32}$$

where $p=p(\boldsymbol{\varphi},t|\boldsymbol{\varphi}_0,t_0)$ is the transition probability density and $\mathbf{J}$ is the hereby defined probability current density. We note that in our case $\mathbf{C}^{-1}(\mathbf{C}^{-1})^{\mathrm{T}}=(\mathbf{C}^{\mathrm{T}}\mathbf{C})^{-1}$.

A stationary solution of Eq. (32), means $\mathbf{J}=$const. For natural boundaries where the probability current vanishes we have

$$J_i = \sum_j\left[-\frac{1}{\tau}\delta_{ij}\varphi_j p_s - \frac{\sigma^2}{2\tau^2\Delta x}(\mathbf{C}^{\mathrm{T}}\mathbf{C})^{-1}_{ij}\frac{\partial}{\partial\varphi_j}p_s\right]=0. \tag{33}$$

From Eq. (33) we get

$$\frac{\partial}{\partial\varphi_l}\ln p_s = \sum_j\left[-\frac{2\tau\Delta x}{\sigma^2}(\mathbf{C}^{\mathrm{T}}\mathbf{C})_{lj}\varphi_j\right]. \tag{34}$$

Now the nonvanishing elements of $\mathbf{C}^{\mathrm{T}}\mathbf{C}$ can be computed from Eq. (30) as

$$(\mathbf{C}^{\mathrm{T}}\mathbf{C})_{i,i}=c_0^2+c_1^2, \quad (\mathbf{C}^{\mathrm{T}}\mathbf{C})_{i,i\pm1}=c_0c_1. \tag{35}$$

As the right-hand side of Eq. (34) is a gradient ($\mathbf{C}^{\mathrm{T}}\mathbf{C}$ is symmetric), the potential conditions are fulfilled and a simple integration gives

$$p_s(\boldsymbol{\varphi}) = \mathcal{N}\exp\left[-\frac{\tau\Delta x}{\sigma^2}\boldsymbol{\varphi}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{C}\boldsymbol{\varphi}\right], \tag{36}$$

where $\mathcal{N}$ is the normalization factor. $\mathbf{C}^{\mathrm{T}}\mathbf{C}$ is an oscillation matrix [28] with the positive eigenvalues

$$\Lambda_j = 1+2\left(\frac{\lambda}{\Delta x}+\frac{\lambda^2}{\Delta x^2}\right)\left[1-\cos\left(\frac{\pi j}{N+1}\right)\right]. \tag{37}$$

Thus the stationary solution can be normalized, $\mathcal{N}=\Pi_{j=1}^{N}\sigma(\pi\tau\Delta x\Lambda_j)^{-1/2}$, and the stationary probability density is indeed the zero mean Gaussian distribution (36).

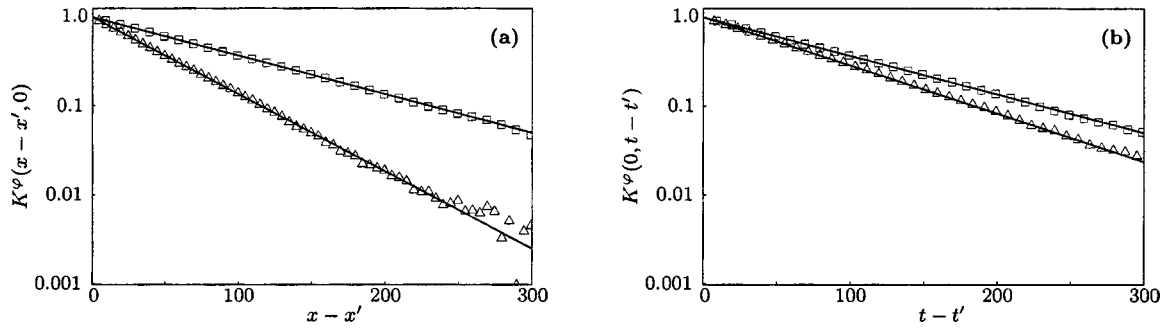

FIG. 1. Autocorrelation of the generalized OUP in $1+1$ dimensions normalized by the variance $\tilde{\sigma}^2/4\lambda\tau$. Comparison of analytical and numerical results for (a) fixed temporal ($t=t'=100$) and (b) fixed spatial ($x=x'=100$) argument. The lines show the analytic results from Eq. (27), the symbols are the results of simulations (squares, $\lambda=\tau=100$; triangles, $\lambda=50$, $\tau=80$). Stationarity was ensured imposing the corresponding initial and boundary processes, see text. Averages over 10^5 realizations ($N=1000$, $\tilde{\sigma}=1$, $\Delta x=\lambda/100$, $\Delta t=\tau/100$).

C. Comparison with numerics

We compare the analytically given autocorrelation with numerically generated data obtained with the discrete integral algorithm provided by Eq. (20). Figure 1 shows a good agreement for fixed temporal and fixed spatial argument, respectively, imposing initial and boundary conditions which ensure stationarity and homogeneity as described above.

We also determined the mean square deviation of the variance of averages over 10^5 independent realizations which is governed by the χ^2 distribution. The variance was always found within a 80% confidence interval.

IV. THE APPROACH OF GSR IN 1+1 DIMENSIONS

The above proposed generalization of the OUP has *by construction* autocorrelations decaying exponentially in both space and time. This is in contrast with the spatiotemporal correlated noise proposed by GSR [13]. Since the autocorrelation for $1+1$ dimensions in real space was not explicitly calculated in the previous literature, below we consider this case. Again, we derive the result in a continuous approach and in a spatially discretized scheme and compare the analytical results with numerical data. The autocorrelation in real space for spatial dimension $d\geqslant 2$ is evaluated in a different context in Ref. [29]. In reciprocal space, the result is given for general d in Refs. [12,19], cf. also Ref. [20].

A. Continuous approach

We start with the Fourier transform of Eq. (1) in $d=1$, which reads

$$\tau\frac{\partial}{\partial t}\varphi(k,t)=-c(k)\varphi(k,t)+\xi(k,t), \tag{38}$$

where $c(k)=1+\lambda^2k^2$ and $\xi(k,t)$ is the Fourier transformed white noise with autocorrelation

$$K^{\xi}(k,t,k',t')=2\pi\sigma^2\delta(k+k')\,\delta(t-t'). \tag{39}$$

Equation (38) defines an OUP for each k. It has the general solution

$$\varphi(k,t)=e^{-c(k)t/\tau}\varphi(k,0)+\frac{1}{\tau}e^{-c(k)t/\tau}\int_0^t ds\,\xi(k,s)e^{c(k)s/\tau}. \tag{40}$$

Stationarity and homogeneity are ensured if the initial values have the autocorrelation

$$K^{\varphi}(k,0;k',0)=\frac{\sigma^2}{2\tau}\frac{2\pi}{c(k)}\delta(k+k'), \tag{41}$$

as for the Fourier transform of a spatial OUP with variance $\sigma^2/4\tau\lambda$. The autocorrelation function in the stationary and homogeneous case is

$$K^{\varphi}(k,t;k',t')=\frac{\sigma^2}{2\tau}\frac{2\pi}{c(k)}\delta(k+k')e^{-c(k)|t-t'|/\tau}, \tag{42}$$

which is up to constant factors in accordance with Refs. [12,19]. Inverse Fourier transform gives

$$K^{\varphi}(x-x',t-t')=\frac{\sigma^2}{2\tau}\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\frac{1}{c(k)}\times e^{-c(k)|t-t'|/\tau-ik(x-x')}, \tag{43}$$

where we introduced $K^{\varphi}(x-x',t-t')=K^{\varphi}(x,x';t,t')$. To calculate the integral on the right-hand side of Eq. (43) we introduce $\tilde{k}=\lambda k$ and $\tilde{c}(\tilde{k})=1+\tilde{k}^2=c(k)$ and note that K^{φ} depends only on $\rho=(x-x')/\lambda$ and $s=|t-t'|/\tau$. For $s=0$ one readily obtains $K^{\varphi}(\rho,0)=\sigma^2/4\tau\lambda e^{-|\rho|}$. In the general case we observe that the derivative of Eq. (43) with respect to s reduces to the Fourier transform of a Gaussian

$$\frac{\partial K^{\varphi}(\rho,s)}{\partial s}=-\frac{\sigma^2}{2\tau\lambda}\frac{1}{2\pi}\int_{-\infty}^{\infty}d\tilde{k}e^{-\tilde{c}(\tilde{k})s-i\tilde{k}\rho}=-\frac{\sigma^2}{4\tau\lambda}\frac{1}{\sqrt{\pi s}}e^{-s-\rho^2/4s}. \tag{44}$$

Integration with respect to s gives

$$K^{\varphi}(\rho,s) = -\frac{\sigma^2}{4\tau\lambda}\frac{1}{\sqrt{\pi}}\int_{s_0}^{s} ds' \frac{1}{\sqrt{s'}} e^{-s'-\rho^2/4s'}$$
$$= -\frac{\sigma^2}{4\tau\lambda}\frac{2}{\sqrt{\pi}}\int_{\sqrt{s_0}}^{\sqrt{s}} dy e^{-y^2-\rho^2/4y^2}$$
$$= -\frac{\sigma^2}{4\tau\lambda}\frac{1}{2}\left[e^{\rho}\mathrm{erf}\left(y+\frac{\rho}{2y}\right) + e^{-\rho}\mathrm{erf}\left(y-\frac{\rho}{2y}\right)\right]_{\sqrt{s_0}}^{\sqrt{s}}, \tag{45}$$

where $\mathrm{erf}(x) = 2/\sqrt{\pi}\int_0^x dt\, e^{-t^2}$ is the error function. In the limit $s\to\infty$ the autocorrelation should vanish, hence

$$K^{\varphi}(\rho,s) = -\lim_{s_0\to\infty}\frac{\sigma^2}{8\tau\lambda}$$
$$\times\left\{ e^{\rho}\left[\mathrm{erf}\left(\sqrt{s}+\frac{\rho}{2\sqrt{s}}\right) - \mathrm{erf}\left(\sqrt{s_0}+\frac{\rho}{2\sqrt{s_0}}\right)\right]\right.$$
$$\left. + e^{-\rho}\left[\mathrm{erf}\left(\sqrt{s}-\frac{\rho}{2\sqrt{s}}\right) - \mathrm{erf}\left(\sqrt{s_0}-\frac{\rho}{2\sqrt{s_0}}\right)\right]\right\}. \tag{46}$$

The limit $s_0\to\infty$ should be carefully taken. If we are interested in the limit $\lambda\to 0$ or in the asymptotics for large ρ, the corresponding operation has to be done before $s_0\to\infty$. The limit $\lambda\to 0$ of Eq. (46) leads to

$$K^{\varphi}(s,x-x') = \frac{\sigma^2}{2\tau}\delta(x-x')e^{-s} \tag{47}$$

as expected. Evaluating first the limit $\tau\to 0$ of Eq. (46) results in

$$K^{\varphi}(t-t',\rho) = \frac{\sigma^2}{4\lambda}\delta(t-t')(1+|\rho|)e^{-|\rho|}. \tag{48}$$

Independent on the order of the limits, we obtain for both λ and $\tau\to 0$ the result for spatiotemporal Gaussian white noise $K^{\varphi} = \sigma^2\delta(x-x')\delta(t-t')$ which can be also directly inferred from Eq. (1).

The asymptotics for $\rho\gg 1$ and $s=\mathrm{const}\neq 0$ and, alternatively, for $s\gg 1$ and $\rho=\mathrm{const}$, is obtained employing $\mathrm{erf}(z) \sim \pm 1 - 1/\sqrt{\pi}z e^{-z^2}$ for $z\to\pm\infty$, cf., e.g., Ref. [30], as

$$K^{\varphi}(\rho,s) \sim \frac{\sigma^2}{4\tau\lambda}\sqrt{\frac{s}{\pi s-\rho^2/4s}}e^{-\rho^2/4s}e^{-s}. \tag{49}$$

For $s\ll 1$ and $\rho=\mathrm{const}\neq 0$ one obtains, from Eq. (46), after first doing $s_0\to\infty$ and employing again the asymptotics of $\mathrm{erf}(z)$,

$$K^{\varphi}(\rho,s) \sim \frac{\sigma^2}{4\tau\lambda}\left\{ e^{-|\rho|} + \sqrt{\frac{s}{\pi s-\rho^2/4s}}e^{-\rho^2/4s}e^{-s}\right\}, \tag{50}$$

where the second term on the right-hand side vanishes for $s\to 0$.

For $\rho\ll 1$ and $s=\mathrm{const}\neq 0$ expanding $\mathrm{erf}(\sqrt{s}\pm\rho/2\sqrt{s})$ and $e^{\pm\rho}$ one obtains from Eq. (46), independent of the order of the limits, up to second order in ρ,

$$K^{\varphi}(\rho,s) \approx \frac{\sigma^2}{8\tau\lambda}\left\{ (1-\mathrm{erf}\sqrt{s})(2+\rho^2) - \frac{\rho^2}{\sqrt{\pi s}}e^{-s}\right\}. \tag{51}$$

The limit $\tau\to 0$ leads to $K^{\varphi}(\rho,s) = \sigma^2/(4\lambda)\delta(t-t')(1-\rho^2/2)$ in accordance with the expansion of Eq. (48) for small ρ. Using the asymptotics $1-\mathrm{erf}\sqrt{s}\sim e^{-s}(1-1/2s)/\sqrt{\pi s}$ for large s one obtains $K^{\varphi} \sim \sigma^2/(4\tau\lambda\sqrt{\pi s})e^{-s}(1-\rho^2/4s)$ which agrees with the expansion of Eq. (49) for small ρ.

The autocorrelation function should solve the equation obtained by multiplying Eq. (1) with $\varphi(x',t')$ and averaging,

$$\tau\frac{\partial}{\partial t}K^{\varphi}(x-x',t-t') = -(1-\lambda^2\Delta)K^{\varphi}(x-x',t-t')$$
$$+\Theta(t'-t)\frac{\sigma^2}{2\tau\lambda}\frac{1}{\sqrt{\pi s}}e^{-s-\rho^2/4s}, \tag{52}$$

which is fulfilled by Eq. (46). In the limit $\lambda\to 0$ the inhomogeneity reduces to that of Eq. (7) multiplied by $\delta(x-x')$ as it should be. In the limit $\tau\to 0$ the inhomogeneity of Eq. (52) becomes $\sigma^2/(2\lambda)\delta(t-t')e^{-|\rho|}$ [31] which can be also directly derived.

Sancho *et al.* [22] claimed that the decay of correlations is exponentially dominated in both space and time. The above results show that this is generally not the case for $d=1$; see Ref. [33].

B. Spatially discretized scheme

García-Ojalvo *et al.* [13] calculated the autocorrelation of the GSR process for $d=2$ in discrete space. Here we repeat the procedure in $d=1$ to compare it with the continuous case. The spatially discretized version of Eq. (1) reads

$$\tau\frac{\partial}{\partial t}\varphi_j(t) = -\varphi_j(t) + \lambda^2\Delta\varphi_j(t) + \xi_j(t), \tag{53}$$

where the Euler discretization of the Laplacian is

$$\Delta\varphi_j(t) = \frac{1}{\Delta x^2}[\varphi_{j+1}(t) - 2\varphi_j(t) + \varphi_{j-1}(t)]. \tag{54}$$

In discrete space we have to rescale the white noise according to

$$\langle \xi_j(t)\xi_{j'}(t')\rangle = \frac{\sigma^2}{\Delta x}\delta_{j,j'}\delta(t-t'). \tag{55}$$

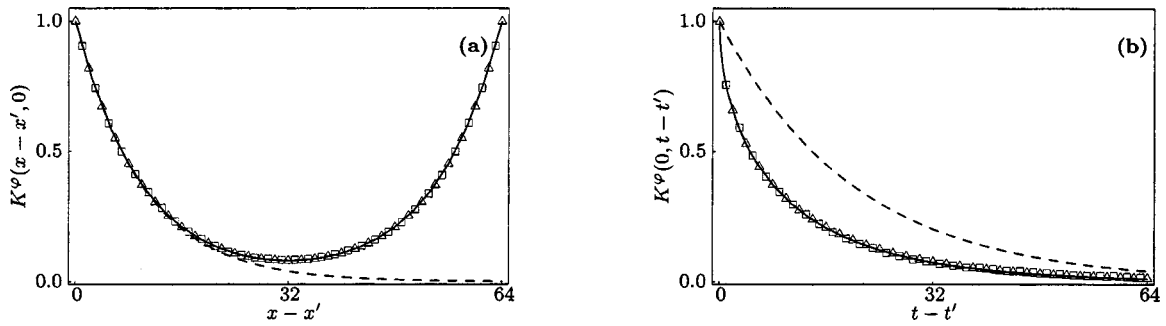

FIG. 2. Autocorrelation (normalized by the variance) of the GSR process in $1+1$ dimensions. Comparison of simulations in finite Fourier space (periodic boundary conditions) with analytical results in discrete and continuous space. Squares denote results of simulations of Eq. (53) with additional Euler discretization of time. Triangles are the analytical result in discrete space given by Eq. (62). (a) shows the spatial dependence after a transient period $t=t'=1000$. The solid line is the plot of $K^\varphi(\rho,0)=\sigma^2/4\tau\lambda(e^{-\rho}+e^{\rho-N\Delta x/\lambda})$ which solves Eq. (52) for periodic boundary conditions at $t=t'$. The dashed line is the plot of the result for the infinite system, $K^\varphi(\rho,0)=\sigma^2/4\tau\lambda e^{-\rho}$, which coincides with the generalized OUP, cf. Eq. (13). (b) shows the temporal dependence for $x=x'=32$. Here, the solid line is the plot of $K^\varphi(0,s)=\sigma^2/4\tau\lambda(1-\mathrm{erf}\sqrt{s})$, cf. Eq. (51). The dashed line is again the result for the generalized OUP, cf. Eq. (13). Averages over 10^5 realizations ($N=64$, $\tilde{\sigma}=1$, $\Delta x=\Delta t=1$, $\lambda=10$, and $\tau=20$).

Again, as in the continuous case, we Fourier transform, solve the decoupled equations, and calculate the autocorrelation function. We define the discrete Fourier transform on the spatial lattice as

$$\varphi_\mu(t)=\Delta x\sum_{j=0}^{N-1} e^{i(2\pi/N)\mu j}\varphi_j(t). \tag{56}$$

Hence the inverse Fourier transform is given by

$$\varphi_j(t)=\frac{1}{N\Delta x}\sum_{\mu=0}^{N-1} e^{-i(2\pi/N)\mu j}\varphi_\mu(t). \tag{57}$$

Greek indices are used in Fourier space and Latin indices in real space. The indices run from 0 to $N-1$ in both real and Fourier space; due to periodic boundaries, $-\mu$ has to be interpreted as $N-\mu$. Now we can Fourier transform Eq. (53),

$$\tau\frac{\partial}{\partial t}\varphi_\mu=-c_\mu\varphi_\mu(t)+\xi_\mu(t), \tag{58}$$

where

$$c_\mu=1-2\frac{\lambda^2}{\Delta x^2}\left[\cos\left(\frac{2\pi\mu}{N}\right)-1\right]. \tag{59}$$

The autocorrelation function of the Fourier transformed white noise is

$$\langle\xi_\mu(t)\xi_{\mu'}(t')\rangle=\sigma^2N\Delta x\,\delta_{\mu,-\mu'}\delta(t-t'). \tag{60}$$

As in continuous space, Eq. (58) defines an Ornstein-Uhlenbeck process with autocorrelation time τ/c_μ for each μ. The stationary autocorrelation can be computed in complete analogy to continuous space as the inverse Fourier transform of

$$K^\varphi_{\mu,\mu'}(t-t')=\frac{\sigma^2}{2\tau}\frac{N\Delta x}{c_\mu}\delta_{\mu,-\mu'}e^{-c_\mu|t-t'|/\tau}. \tag{61}$$

Hence the stationary autocorrelation in discrete space is

$$K_{j-j'}(t-t')=\frac{\sigma^2}{2\tau}\frac{1}{N\Delta x}\sum_{\mu=0}^{N-1}\frac{1}{c_\mu}e^{-c_\mu|t-t'|/\tau-i(2\pi/N)\mu(j-j')}. \tag{62}$$

Since $c_\mu=c_{N-\mu}$, the imaginary part of the sum vanishes. For $|t-t'|/\tau\gg1$ the autocorrelation is dominated by the first term $e^{-|t-t'|/\tau}$ in the sum ($\mu=0$). However, this is not so for $|t-t'|\approx\tau$, cf. Fig. 2(b).

Observing $N\Delta x=L$, L being the system size, and identifying $k=2\pi\mu/L$, we have in the limit $\Delta x\to0$ the correspondence $c_\mu\to c(k)=1+\lambda^2k^2$. Hence Eq. (61) corresponds to Eq. (42) and Eq. (62) to Eq. (43) after doing the limit $L\to\infty$ in an appropriate way.

C. Comparison with numerics

The initial conditions for a stationary field in Fourier space were chosen as independent Gaussian random numbers with variance $\sigma^2(N\Delta x)^2/\tau c_\mu$ for each μ. The spatial autocorrelation was computed using the correlation theorem (cf., e.g., Ref. [34]) valid for weak stationary ergodic processes

$$\mathcal{F}[\langle g(x_0)g(x_0+x)\rangle]=\mathcal{F}[g(x)]\mathcal{F}[-g(x)], \tag{63}$$

where $\mathcal{F}[g(x)]$ denotes the Fourier transform of $g(x)$. The procedure is faster in numerical simulations and gives the same results as the real-space approach; moreover, the inverse Fourier transform can be avoided if one is interested only in spatial correlations.

Figure 2 compares numerical and analytical results for the GSR process in $1+1$ dimensions.

Simulations in real space give results that coincide with those in Fourier space; we refrain from demonstrating this here. A simulation in real space has the disadvantage that the

maximal possible time step Δt is restricted by $\Delta t < \tau \Delta x^2 / 4\lambda^2$, which enforces a small Δt, otherwise the discrete Eq. (53) loses stability; cf. Ref. [13]. Simulations employing periodic boundary conditions or stochastic boundaries with exponentially decaying autocorrelation indicate that the GSR process does not depend on the boundary conditions if we are not too close to the boundaries.

V. CONCLUSIONS

We introduced in $1+1$ dimensions a spatiotemporal stochastic process with an autocorrelation exponentially decaying both in space and time, thus being a generalization of the OUP. An analogous generalization to higher spatial dimensions, although formally possible, seems physically not meaningful, as the autocorrelation function would factorize in the spatial variables. Instead, for higher dimensions the spatial part of the autocorrelation function should only depend on the spatial distance. In addition, the construction of a higher-dimensional process with exponentially decaying autocorrelation should relay to that of the common OUP. For the relativistic OUP [35] a similar situation appears already in $1+1$ dimensions: The spatiotemporal autocorrelation is not allowed to factorize as space and time are intimately connected.

The situation resembles that of the checkerboard process in $1+1$ dimensions [36–39] driven by a velocity changing randomly the sign which is modeled by the simplest discrete process with exponentially decaying autocorrelation, the dichotomous Markovian process. The checkerboard process is related to the Dirac equation or the Klein-Gordon equation in $d=1$. Also, there the generalization to higher spatial dimensions meets nontrivial difficulties [37,38].

Note that there exists a numerical method, viz., the Fourier filter method, which allows one to generate noise of arbitrary autocorrelation provided its Fourier transform exists [2]. In this paper we are however interested in a constructive approach close to that for the OUP employing stochastic differential equations.

ACKNOWLEDGMENTS

The work was partially supported by the DFG (Grant No. Be 1417/3). A.T. acknowledges support by the Studienstiftung des Deutschen Volkes. Thanks are due to Dr. Markus Brede for a valuable remark.

[1] W. Horsthemke and R. Lefever, *Noise-Induced Transitions* (Springer, Berlin, 1984).
[2] J. García-Ojalvo and J.M. Sancho, *Noise in Spatially Extended Systems* (Springer, Berlin, 1999).
[3] K. Honda, Fractals **4**, 331 (1996).
[4] K. Honda, Phys. Rev. E **55**, R1235 (1997).
[5] C.-H. Lam and F.G. Shin, Phys. Rev. E **57**, 6506 (1998).
[6] J. García-Ojalvo and J.M. Sancho, Phys. Rev. E **49**, 2769 (1994).
[7] W. Pesch and U. Behn, in *Evolution of Spontaneous Structures in Dissipative Continuous Systems*, edited by F.H. Busse and S.C. Müller (Springer, Berlin, 1998), p. 335.
[8] U. Behn, A. Lange, and T. John, Phys. Rev. E **58**, 2047 (1998).
[9] T. John, R. Stannarius, and U. Behn, Phys. Rev. Lett. **83**, 749 (1999).
[10] H. Wang and Q. Ouyang, Phys. Rev. E **65**, 046206 (2002).
[11] H. Busch and F. Kaiser, Phys. Rev. E **67**, 041105 (2003).
[12] M.O. Vlad, M.C. Mackey, and J. Ross, Phys. Rev. E **50**, 798 (1994).
[13] J. García-Ojalvo, J.M. Sancho, and L. Ramirez-Piscina, Phys. Rev. A **46**, 4670 (1992).
[14] G.E. Uhlenbeck and L.S. Ornstein, Phys. Rev. **36**, 823 (1930).
[15] S. Chandrasekhar, Rev. Mod. Phys. **15**, 1 (1943).
[16] M.C. Wang and G.E. Uhlenbeck, Rev. Mod. Phys. **17**, 323 (1945).
[17] I. Gyöngy and E. Pardoux, Probab. Theory Relat. Fields **94**, 413 (1993).
[18] C.W. Gardiner, *Handbook of Stochastic Methods*, 2nd ed. (Springer, Berlin, 1985).
[19] P.-M. Lam and D. Bagayoko, Phys. Rev. E **48**, 3267 (1993).
[20] There are a few mistakes in Ref. [19] performing the inverse Fourier transform of the correct result in reciprocal space. For $d=2$ the term $(1+\lambda^2 k^2)^{-1}$ in Eq. (14) is expanded in a geometric series without concern to the convergence condition $|\lambda k|<1$. Similarly, in Eq. (15) the convergence condition $|t-t'|/\tau > |\mathbf{r}-\mathbf{r}'|/\lambda$ for summation of the geometric series is not observed. Formula (6.631.4) of Gradshteyn and Ryzhik [21] is not applicable to treat the integral in Eq. (14) which involves only the zero-order Bessel function. A correct evaluation can be found in Ref. [29]. For $d=3$ a factor $|\mathbf{r}-\mathbf{r}'|^{-1}$ is lacking in Eqs. (17) and (18) which appears by integrating over the colatitude in spherical coordinates; one immediately notes that on the right-hand side of Eq. (18) a factor of dimension $(\text{length})^{-1}$ is missing.
[21] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, London, 1994).
[22] J.M. Sancho, J. García-Ojalvo, and H. Guo, Physica D **113**, 331 (1998).
[23] J.L. Doob, Ann. Math. **43**, 351 (1942).
[24] R.F. Fox, I.R. Gatland, R. Roy, and G. Vemuri, Phys. Rev. A **38**, 5938 (1988).
[25] D.T. Gillespie, Phys. Rev. E **54**, 2084 (1996); see also D.T. Gillespie, *Markov Processes: An Introduction for Physical Scientists* (Academic Press, New York, 1992).
[26] X.-Q. Wei, L. Cao, and D.-J. Wu, Phys. Lett. A **207**, 338 (1995).
[27] R. Mannella and V. Palleschi, Phys. Rev. A **40**, 3381 (1989).
[28] F.R. Gantmacher and M.G. Krein, *Oscillation Matrices and Kernels and Small Vibrations of Mechanical Systems* (AMS Chelsea, New York, 2002).
[29] A. Traulsen and U. Behn (unpublished).
[30] *Handbook of Mathematical Functions*, Nat. Bur. Stand. Appl. Math. Ser. No. 55, edited by M. Abramowitz and I.A. Stegun (U.S. GPO, Washington, D.C., 1964).

[31] Observe that $1/(2\tau\sqrt{\pi s})e^{-s-\rho^2/4s} \to \delta(t-t')e^{-|\rho|}$ in the limit $\tau \to 0$, cf. [32] formula (314.9a).

[32] *Integraltafel, Zweiter Teil: Bestimmte Integrale*, edited by W. Gröbner and N. Hofreiter, 2nd ed. (Springer, Wien, 1958).

[33] For $t > t'$ an exponential decay of the temporal correlation with $e^{-|t-t'|/\tau}$ would imply a constant spatial correlation, which is clearly unphysical. Similarly, a spatial decay of the autocorrelation with $e^{-|x-x'|/\lambda}$ would lead to a temporally constant autocorrelation.

[34] W.H. Press, S.A. Teukolsky, W.T. Vetterling, and B.P. Flannery, *Numerical Recipes in C*, 2nd ed. (Cambridge University Press, New York, 1993).

[35] C. Barbachoux, F. Debbasch, and J.P. Rivet, Eur. Phys. J. B **19**, 37 (2001).

[36] R.P. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965), p. 35.

[37] B. Gaveau, T. Jacobson, M. Kac, and L.S. Schulman, Phys. Rev. Lett. **53**, 419 (1984).

[38] M. Ibison, Chaos, Solitons Fractals **10**, 1 (1999).

[39] A.V. Plyukhin and J. Schofield, Phys. Rev. E **64**, 037101 (2001).